

An Explicit Representation of Solutions of Forward SDE's with Reflections via White Noise Analysis

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Abstract. In this paper we derive an explicit representation for strong solutions of a class of forward stochastic differential equations with reflections (FSDER's). Our approach relies on techniques from white noise analysis. The results obtained in this paper are relevant for the construction of strong solutions of FSDER's with discontinuous coefficients.

1 Introduction

In this paper we aim at establishing a representation formula for strong solutions of FSDER's by employing methods from white noise analysis. Adopting the ideas in [5] and [7] we mention that the results of this paper can be used to construct strong solutions of a class of FSDER's with non-Lipschitz coefficients. See [8].

The plan of the paper is the following. The first Section is based on the material presented in [1], [9], [10], [4] and [2]. Its objective is to introduce briefly some main concepts of the Gaussian white noise theory. In Section 2 some relevant results from the FSDER theory are reviewed. Finally, Section 3 provides a derivation of our main result.

2 Framework

We begin with giving a construction of Hida distributions on \mathbb{R}^d . Let A be a positive self-adjoint operator on $L^2([0, T])$ with $\text{Spec}(A) > 1$ and a fixed time

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horizon $0 < T < \infty$. Assume that A^{-r} is a Hilbert-Schmidt operator for some $r > 0$. Then the set of eigenfunctions of A forms a complete orthonormal basis $\{e_j\}_{j \geq 0}$ of $L^2([0, T])$ in $Dom(A)$ with eigenvalues $\lambda_j > 0$, $j \geq 0$ such that

$$Ae_j = \lambda_j e_j \quad (1)$$

for all $j \geq 0$. We assume that $1 < \lambda_0 \leq \lambda_1 \leq \dots \rightarrow \infty$, each e_j is continuous on $[0, T]$ and that there exists an open covering $[0, T] = \cup_\lambda O_\lambda$ and $\alpha(\lambda) \geq 0$ such that

$$\sup_{j \geq 0} \lambda_j^{-\alpha(\lambda)} \sup_{t \in O_\lambda} |e_j(t)| < \infty. \quad (2)$$

Following the notation in [9], we denote by $\mathcal{S}([0, T])$ the standard countably Hilbertian space constructed from $(L^2([0, T]), A)$. $\mathcal{S}([0, T])$ is a nuclear space which is contained in $L^2([0, T])$, with a topological dual denoted by $\mathcal{S}'([0, T])$ being a conuclear space. Denote by $\mathcal{B}(\mathcal{S}'([0, T]))$ the Borel σ -algebra of $\mathcal{S}'([0, T])$. One can now apply the Bochner-Minlos theorem to find a unique probability measure π on $\mathcal{B}(\mathcal{S}'([0, T]))$ such that

$$\int_{\mathcal{S}'([0, T])} e^{i\langle \omega, \phi \rangle} \pi(d\omega) = e^{-\frac{1}{2} \|\phi\|_{L^2([0, T])}^2} \quad (3)$$

for $\phi \in \mathcal{S}([0, T])$, where $\langle \omega, \phi \rangle$ denotes the action of $\omega \in \mathcal{S}'([0, T])$ on $\phi \in \mathcal{S}([0, T])$.

We now define the *d-dimensional white noise probability measure* μ given by the product measure

$$\mu = \otimes_{i=1}^d \pi. \quad (4)$$

on the measurable space

$$(\mathcal{S}', \mathcal{B}) := \left(\prod_{i=1}^d \mathcal{S}'([0, T]), \otimes_{i=1}^d \mathcal{B}(\mathcal{S}'([0, T])) \right) \quad (5)$$

For $\omega = (\omega_1, \dots, \omega_d) \in \mathcal{S}'$ and $\phi = (\phi^{(1)}, \dots, \phi^{(d)}) \in (\mathcal{S}([0, T]))^d$ we introduce the exponential functional

$$\tilde{e}(\phi, \omega) = \exp \left(\langle \omega, \phi \rangle - \frac{1}{2} \|\phi\|_{L^2([0, T]; \mathbb{R}^d)}^2 \right), \quad (6)$$

where $\langle \omega, \phi \rangle := \sum_{i=1}^d \langle \omega_i, \phi_i \rangle$. Let us denote by $((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$ the n -th completed symmetric tensor product of $(\mathcal{S}([0, T]))^d$ with itself. One can observe that $\tilde{e}(\phi, \omega)$ is holomorphic in ϕ around zero, which implies that

there exist generalized Hermite polynomials $H_n(\omega) \in \left(((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n} \right)'$ such that

$$\tilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \langle H_n(\omega), \phi^{\otimes n} \rangle \quad (7)$$

for ϕ in a certain neighbourhood of zero in $(\mathcal{S}([0, T]))^d$. One can also observe that

$$\left\{ \langle H_n(\omega), \phi^{(n)} \rangle : \phi^{(n)} \in \left((\mathcal{S}([0, T]))^d \right)^{\widehat{\otimes} n}, n \in \mathbb{N}_0 \right\} \quad (8)$$

is a total set of $L^2(\mu)$. Moreover, one can show that the following orthogonality relation is valid for all $n, m, \phi^{(n)} \in \left((\mathcal{S}([0, T]))^d \right)^{\widehat{\otimes} n}, \psi^{(m)} \in \left((\mathcal{S}([0, T]))^d \right)^{\widehat{\otimes} m}$:

$$\int_{\mathcal{S}^1} \langle H_n(\omega), \phi^{(n)} \rangle \langle H_m(\omega), \psi^{(m)} \rangle \mu(d\omega) = \delta_{n,m} n! \left(\phi^{(n)}, \psi^{(n)} \right)_{L^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})}, \quad (9)$$

where $\delta_{n,m} = 1$ if $n = m$ and 0 otherwise. The latter expression implies that $\phi^{(n)} \mapsto \langle H_n(\omega), \phi^{(n)} \rangle$ has a unique extension to $L^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ for ω a.e.

The functional $\langle H_n(\omega), \phi^{(n)} \rangle$ can be regarded as a n -fold iterated stochastic integral of functions $\phi^{(n)} \in L^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ with respect to a d -dimensional Wiener process $B_t = (B_t^1, \dots, B_t^d)$ defined on the white noise space

$$(\Omega, \mathcal{F}, \mu) = (\mathcal{S}^1, \mathcal{B}, \mu) \quad (10)$$

Let us now denote by $\widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ the space of square integrable symmetric functions $f(x_1, \dots, x_n) \in (\mathbb{R}^d)^{\otimes n}$. An important consequence of (7), (8) and (9) is that square integrable functionals of B_t admit a Wiener-Itô chaos representation. The latter can be considered an infinite-dimensional Taylor expansion, i.e. for all $F \in L^2(\mu)$ there exist unique kernels $\phi^{(n)} \in \widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ such that

$$F(\omega) = \sum_{n \geq 0} \langle H_n(\omega), \phi^{(n)} \rangle \quad (11)$$

for ω a.e. The Wiener-Itô chaos expansion (11) can now be used to define the Hida stochastic test function and distribution space.

We construct the *Hida stochastic test function space* (\mathcal{S}) through a second quantization argument, that is we define (\mathcal{S}) to be the space of all

$f = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle \in L^2(\mu)$ such that

$$\|f\|_{0,p}^2 := \sum_{n \geq 0} n! \left\| \left((A^d)^{\otimes n} \right)^p \phi^{(n)} \right\|_{L^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})}^2 < \infty \quad (12)$$

for all $p \geq 0$, where $A^d = (A, \dots, A)$. The space (\mathcal{S}) is a nuclear Fréchet algebra with respect to multiplication of functions and its topology is given by the seminorms $\|\cdot\|_{0,p}$, $p \geq 0$. In particular, one can use (7) to show that

$$\tilde{e}(\phi, \omega) \in (\mathcal{S}) \quad (13)$$

for all $\phi \in (\mathcal{S}([0, T]))^d$.

We define the *Hida stochastic distribution space* $(\mathcal{S})^*$ as the topological dual of (\mathcal{S}) and get the Gelfand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*. \quad (14)$$

An important property of the Hida distribution space $(\mathcal{S})^*$ is that it contains the *white noise* of the coordinates of the d -dimensional Wiener process B_t . That is the time derivatives of the Wiener process

$$W_t^i := \frac{d}{dt} B_t^i, \quad i = 1, \dots, d \quad (15)$$

belong to $(\mathcal{S})^*$.

We now introduce another key concept of the white noise theory, namely the S -transform. The S -transform of $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$, is defined through the dual pairing

$$S(\Phi)(\phi) = \langle \Phi, \tilde{e}(\phi, \omega) \rangle \quad (16)$$

for $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$, where $(\mathcal{S}_{\mathbb{C}}([0, T]))$ is the complexification of $\mathcal{S}([0, T])$. The S -transform is a monomorphism from $(\mathcal{S})^*$ to \mathbb{R} . In particular, if

$$S(\Phi) = S(\Psi) \text{ for } \Phi, \Psi \in (\mathcal{S})^*$$

then

$$\Phi = \Psi.$$

One can also show that

$$S(W_t^i)(\phi) = \phi^i(t), \quad i = 1, \dots, d \quad (17)$$

for $\phi = (\phi^{(1)}, \dots, \phi^{(d)}) \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$.

Finally, we introduce the concept of the *Wick product*, which can be considered a tensor algebra multiplication on the Fock space. The Wick product of two distributions $\Phi, \Psi \in (\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi)S(\Psi)(\phi) \quad (18)$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$. As an example of the use of the Wick product one can verify that

$$\langle H_n(\omega), \phi^{(n)} \rangle \diamond \langle H_m(\omega), \psi^{(m)} \rangle = \langle H_{n+m}(\omega), \phi^{(n)} \widehat{\otimes} \psi^{(m)} \rangle \quad (19)$$

for $\phi^{(n)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$, $\psi^{(m)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} m}$. The latter result, as well as (7) imply that

$$\widetilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \quad (20)$$

for $\phi \in (\mathcal{S}([0, T]))^d$. The Wick exponential $\exp^{\diamond}(X)$ of a $X \in (\mathcal{S})^*$ is defined as

$$\exp^{\diamond}(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}, \quad (21)$$

where $X^{\diamond n} = X \diamond \dots \diamond X$, provided the sum on the right hand side converges in $(\mathcal{S})^*$.

3 Forward SDEs with reflections

This section passes in review conditions for the existence and uniqueness of (global strong) solutions of a forward stochastic differential equation with reflections (FSDER). For more information on FSDER's and their applications the reader may consult the excellent book of [6].

A general form of a forward SDE with reflections is the following:

$$\begin{aligned} X(t) &= x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) + \eta(t) \\ x &\in \mathcal{O} \end{aligned} \quad (22)$$

Here \mathcal{O} is a closed convex domain in \mathbb{R}^n ; b and σ are functions of $(t, x, \omega) \in [0, T] \times \mathbb{R}^n \times \Omega$; and $\eta \in BV_{\mathcal{F}}([0, T], \mathbb{R}^m)$, the set of all \mathbb{R}^m -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes η with paths of bounded variation. In the sequel we shall also denote by $L^2_{\mathcal{F}}([0, T], \mathbb{R}^m)$ the space of all $\{\mathcal{F}_t\}_{t \geq 0}$ -progressively measurable processes $X(t)$ in \mathbb{R}^n , such that $\int_0^T \mathbb{E}[\|X(t)\|^2 dt] < \infty$.

Definition 3.1. [6] A pair of continuous, $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $(X, \eta) \in \mathcal{L}_{\mathcal{F}}^2([0, T], \mathbb{R}^m) \times BV_{\mathcal{F}}([0, T], \mathbb{R}^m)$ is called a solution to the FSDER (22) if

- 1) $X(t) \in \mathcal{O}, \forall t \in [0, T], \text{ a.s.};$
- 2) $\eta(t) = \int_0^t 1_{\{X(s) \in \partial \mathcal{O}\}} \gamma(s) d|\eta|(s),$ where $\gamma(s) \in \mathcal{N}_{X(s)}, 0 \leq s \leq t \leq T,$
 $d|\eta|$ -a.e. and $|\eta|(T)$ denotes the total variation of η on $[0, T];$
- 3) equation (22) is satisfied a.s.

Here \mathcal{N}_x is the set of inward normals to \mathcal{O} at x defined as follows:

$$\mathcal{N}_x = \{\gamma : |\gamma| = 1, \langle \gamma, x - y \rangle \leq 0, \forall y \in \mathcal{O}\}, \quad (23)$$

for $x \in \partial \mathcal{O}.$

The issue of existence and uniqueness of a solution of an FSDE with reflections can be related to the so-called Skorohod problem. The latter is defined as follows:

Let the domain \mathcal{O} and a function $\psi \in C([0, T], \mathbb{R}^n)$ with $\psi(0) \in \mathcal{O}$ be given. Find a pair $(\phi, \eta) \in C([0, T], \mathbb{R}^n) \times BV([0, T], \mathbb{R}^n),$ such that

- 1) $\phi(t) = \psi(t) + \eta(t), \forall t \in [0, T],$ and $\phi(0) = \psi(0);$
- 2) $\phi(t) \in \mathcal{O}, \forall t \in [0, T];$
- 3) $|\eta(t)| = \int_0^t 1_{\{\phi(s) \in \partial \mathcal{O}\}} d|\eta|(s);$
- 4) there exists a measurable function $\gamma : [0, T] \rightarrow \mathbb{R}^n,$ such that $\gamma(t) \in \mathcal{N}_{\phi(t)} d|\eta|$ a.s. and $\eta(t) = \int_0^t \gamma(s) d|\eta|(s).$

A pair (ϕ, η) satisfying the above conditions is called a solution of the Skorohod problem. We define a mapping $\Gamma : C([0, T], \mathbb{R}^n) \rightarrow C([0, T], \mathbb{R}^n),$ such that $\Gamma(\psi)(t) = \phi(t), t \in [0, T],$ where (ϕ, η) is the unique solution of the corresponding Skorohod problem. We call Γ the solution mapping of the Skorohod problem, and we call a convex domain $\mathcal{O} \in \mathbb{R}^n$ *regular* if the solution mapping of the corresponding Skrohod problem satisfies the Lipschitz condition:

$$|\Gamma(\psi_1)(\cdot) - \Gamma(\psi_2)(\cdot)|_T^* \leq K |\psi_1(\cdot) - \psi_2(\cdot)|_T^*, \quad (24)$$

where $|\epsilon|_T^*$ denotes the sup-norm on $[0, T]$ for a function $\epsilon \in C([0, T], \mathbb{R}^n),$ and $K > 0.$

As shown in [6], for a regular convex domain \mathcal{O} the FSDER (22) has a unique strong solution provided that the following conditions are satisfied:

- 1) for every fixed $x \in \mathbb{R}^n$, $b(\cdot, x, \cdot)$ and $\sigma(\cdot, x, \cdot)$ are $\{\mathcal{F}_t\}_{t \geq 0}$ -progressively measurable;
- 2) there exists a constant $K > 0$, such that for all $(t, \omega) \in [0, T] \times \Omega$ and $x, y \in \mathbb{R}^n$ it holds that

$$\begin{aligned} |b(t, x, \omega) - b(t, y, \omega)| &\leq K|x - y| \\ |\sigma(t, x, \omega) - \sigma(t, y, \omega)| &\leq K|x - y| \end{aligned} \quad (25)$$

4 White noise representation for FSDEs with reflections

In this section we derive a representation formula for solutions of FSDEs with reflections. Our results stem from the ideas presented in [5] and [7]

Theorem 4.1. *Suppose that $\mathcal{O} \in \mathbb{R}^n$ is a regular, convex domain and assume that the drift term b and the diffusion matrix σ in equation (22) are deterministic. Further, require that σ be independent of the space variable, continuous in time, invertible t a.e., and that $\sigma^{-1}(t)$ be continuously differentiable on $(0, T)$ (with continuous extensions to $[0, T]$). For a fixed initial value $x \in \mathbb{R}^n$ in (22) define the Borel measurable functions $\tilde{b} : [0, T] \times C([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}^n$, and $\Psi : C([0, T], \mathbb{R}^n) \rightarrow C([0, T], \mathbb{R}^n)$ as follows:*

$$\begin{aligned} \tilde{b}(t, \phi) &= \left(\frac{\partial}{\partial t} \sigma^{-1}(t) \right) \sigma(t) \Gamma(\phi)(t) + \\ &\quad + \sigma^{-1}(t) b\left(t, \sigma(t) (\Gamma(\phi)(t) + \sigma^{-1}(0)x)\right) \end{aligned} \quad (26)$$

and

$$\Psi(\phi)(t) = \Gamma(\sigma(\cdot)(\phi + \sigma^{-1}(0)x))(t), \quad (27)$$

where the mapping Γ is the solution mapping of the corresponding Skorohod problem. In addition suppose that the following integrability conditions are satisfied.

$$E \left[\int_0^T \|\Psi(B_\cdot)(t)\|^2 dt \right] < \infty, \quad (28)$$

for $\forall t \in [0, T]$, and

$$E \left[\exp \left(36 \int_0^T \|\tilde{b}(s, B.)\|^2 ds \right) \right] < \infty \quad (29)$$

Denote by $\Psi^i(\phi)(t)$ and $\tilde{b}^i(\phi)$ the i -th component of $\Psi(\phi)(t)$ and $\tilde{b}(\phi)$ for all $\phi \in C([0, T], \mathbb{R}^n)$ and t respectively. Then the unique solution $X_t = (X_t^i)_{i=1, \dots, d}$ of the FSDER (22) is explicitly represented by

$$X_t^i = E_{\hat{\mu}} \left[\Psi^i(\hat{B}(\hat{\omega}))(t) \mathcal{E}_T^\diamond(\tilde{b}) \right], \quad i = 1, \dots, d \quad (30)$$

where the random element $\mathcal{E}_T^\diamond(\tilde{b}) : \tilde{\Omega} \rightarrow (S)^*$ is defined as

$$\begin{aligned} \mathcal{E}_T^\diamond(\tilde{b})(\omega, \hat{\omega}) &= \exp^\diamond \left(\sum_{j=1}^d \int_0^T [\tilde{b}^j(s, \hat{B}(\hat{\omega})) + W_s^j] d\hat{B}_s^j(\hat{\omega}) - \right. \\ &\quad \left. - \frac{1}{2} \int_0^T [\tilde{b}^j(s, \hat{B}(\hat{\omega})) + W_s^j(\omega)]^{\diamond 2} ds \right) \end{aligned} \quad (31)$$

The 4-tuple $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}), (\hat{B}_t)_{t \geq 0}$ is a copy of $(\Omega, \mathcal{F}, \mu), (B_t)_{t \geq 0}$ in (10).

The $E_{\hat{\mu}}$ stands for the Pettis integral of random elements $\Phi : \hat{\Omega} \rightarrow (S)^*$ with respect to the measure $\hat{\mu}$. Here W_t^i in the Wick exponential of (31) also denotes the white noise in the Hida space $(S)^*$. The Wick product \diamond in (31) refers to the measure μ . The ds -integral in (31) is defined in the Pettis sense and the other integrals $\int_0^T \phi(t, \omega) d\hat{B}_s^j(\hat{\omega})$ in (31) are stochastic integrals of predictable $(S)^*$ -valued integrands $\phi(t, \omega)$. (see e.g. [3])

The proof of Theorem 4.1 relies on the following auxiliary result (see e.g. [4, Theorem 13.4]).

Lemma 4.1. *Let (M, \mathcal{B}, ρ) be a measure space. Let a function $\Phi : M \rightarrow (S)^*$ be such that $S(\Phi(\cdot))(\phi)$ is measurable for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$. Denote by $(|\cdot|_p)_{p \geq 0}$ the family of increasing compatible seminorms of $(\mathcal{S}_{\mathbb{C}}([0, T]))^d$. Further, require that there exist $K, a, p \geq 0$ such that*

$$\int_M |S(\Phi(u))(\phi)| \rho(du) \leq K \exp(a |\phi|_p^2)$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$. Then Φ is Pettis integrable and for any $\mathcal{A} \in \mathcal{B}$ and all $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$ we have that

$$S \left(\int_{\mathcal{A}} \Phi(u) \rho(du) \right) (\phi) = \int_{\mathcal{A}} S(\Phi(u))(\phi) \rho(du)$$

Proof of Theorem 4.1. The proof is based on ideas developed in [5] and [7]. Without loss of generality let us confine ourselves to the case when $d = 1$. To simplify notation we set $\Psi := \Psi^1$. Using the properties of the mapping Γ and the regularity of the domain \mathcal{O} one finds that $b^*(t, \phi) = b(t, \Gamma(\phi)(t))$, $\phi \in \mathbb{C}([0, T])$ is a progressively measurable Lipschitz continuous functional (see e.g. [6]). Thus, there exists a unique strong solution \tilde{X}_t to the SDE

$$d\tilde{X}_t = x + \int_0^t b^*(s, \tilde{X}_s) ds + \int_0^t \sigma(s) dB_s \quad (32)$$

On the other hand, invoking the definition of the Skorohod problem it follows that $X_t = \Gamma(\tilde{X})(t)$, $t \in [0, T]$ solves the FSDER (32) uniquely. Set $Y_t = \sigma^{-1}(t)\tilde{X}_t$. Then Ito's Lemma implies that

$$Y_t = \sigma^{-1}(0)x + \int_0^t \tilde{b}(s, Y_s) ds + B_t, \quad t \in [0, T] \quad (33)$$

So it follows from our assumptions that $Y_t \in L^2(\mu)$ for all t .

Suppose that $\pi : [0, T] \times C([0, T]) \rightarrow \mathbb{R}$ is a bounded Borel measurable function. We aim at deriving a representation of the S -transform of $\pi(t, Y)$. Using the definition of the S -transform we see that

$$S(\pi(t, Y)) = E_\mu [\pi(t, Y(\omega + \phi))] \quad (34)$$

for all $\phi \in \mathcal{S}([0, T])$. Girsanov's theorem shows that $Y_t^*(\omega) = Y_t(\omega + \phi)$ satisfies the following SDE

$$dY_t^* = \tilde{b}(t, Y_t^*) + \phi(t)dt + dB_t, \quad Y_0^* = \sigma^{-1}(0)x, \quad 0 \leq t \leq T.$$

Applying Girsanov's theorem to (34) once again yields that

$$S(Y_t)(\phi) = E_{\hat{\mu}} \left[\pi \left(t, \hat{B}_t \right) \mathcal{E}(M_t^\phi) \right] \quad (35)$$

for all $\phi \in \mathcal{S}([0, T])$, where the 4-tuple $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$, $(\hat{B}_t)_{t \geq 0}$ is a copy of $(\Omega, \mathcal{F}, \mu)$, $(B_t)_{t \geq 0}$ and where $\mathcal{E}(M_t^\phi)$ is the usual notation for the Doleans-Dade exponential for the martingale

$$M_s^\phi(\hat{\omega}) = \int_0^s \left(\tilde{b}(t, \hat{B}_t(\hat{\omega})) + \phi(t) \right) d\hat{B}_t(\hat{\omega}).$$

So

$$\begin{aligned} & \mathcal{E}(M_t^\phi) \\ &= \exp \left(\int_0^T \left(\tilde{b}(t, \hat{B}_t) + \phi(t) \right) d\hat{B}_t - \frac{1}{2} \int_0^T \left(\tilde{b}(t, \hat{B}_t) + \phi(t) \right)^2 dt \right) \end{aligned}$$

One can show that relation (35) also holds for $\phi \in \mathcal{S}_{\mathbb{C}}([0, T])$.

On the other hand, we know by (17) suggests that

$$S(W_t)(\phi) = \phi(t)$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. Further we find that

$$\mathcal{E}(M_t^\phi) = \mathcal{E}(M_t^0) \exp \left(\int_0^T \phi(t) d\widehat{B}_t - \frac{1}{2} \int_0^T (\phi(t))^2 dt \right) \exp \left(\int_0^T \tilde{b}(t, \widehat{B}_\cdot) \phi(t) dt \right) \quad (36)$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. The second factor of the right hand side of (36) is the S -transform of the Kubo-Yokoi delta function (see [4, Theorem 13.4]) and it can be written as follows:

$$\mathcal{E}_T^\diamond(0) = \exp^\diamond \left(\int_0^T W_s(\omega) d\widehat{B}_s^j(\widehat{\omega}) - \frac{1}{2} \int_0^T (W_s(\omega))^{\diamond 2} ds \right).$$

The last factor in (36) is G -entire and its S -transform is bounded from above by

$$K^* \exp(a^* |\phi|_0^2),$$

with some constants K^*, a^* and $|\phi|_0 = \|\phi\|_{L_{\mathbb{C}}^2([0, T])}$. Using the characterization theorem of Hida distributions (see [10]) and employing the properties of the S -transform we find that

$$S(\Phi(\widehat{\omega}, \cdot))(\phi) = \pi \left(t, \widehat{B}_\cdot(\widehat{\omega}) \right) \mathcal{E}(M_t^\phi)(\widehat{\omega}),$$

where the map $\Phi : \Omega \times \widehat{\Omega} \longrightarrow (\mathcal{S})^*$ is given by

$$\Phi(\widehat{\omega}, \omega) = \pi \left(t, \widehat{B}_\cdot(\widehat{\omega}) \right) \mathcal{E}_T^\diamond(\tilde{b})(\omega, \widehat{\omega})$$

with $\mathcal{E}_T^\diamond(\tilde{b})$ as in (31). Note that $S(\Phi(\widehat{\omega}, \cdot))(\phi)$ is $\widehat{\omega}$ -measurable for all ϕ .

Now the Hölder inequality and the supermartingale property of Doleans-Dade exponentials imply

$$\begin{aligned} & E_{\widehat{\mu}} [|S(\Phi(\widehat{\omega}, \cdot))(\phi)|] \\ &= E_{\widehat{\mu}} \left[\left| \pi \left(t, \widehat{B}_\cdot \right) \mathcal{E}(M_t^\phi) \right| \right] \\ &\leq K \cdot E_{\widehat{\mu}}^{\frac{1}{2}} \left[\mathcal{E} \left(\int_0^T 2 \left(\tilde{b}(t, \widehat{B}_\cdot) + \operatorname{Re} \phi(t) \right) d\widehat{B}_t \right) \right] \exp(a \int_0^T |\phi(t)|^2 dt) \\ &\leq K \exp(a |\phi|_0^2), \end{aligned}$$

with some constants $a, K \geq 0$. Then Lemma 4.1 above shows that

$$S(\pi(t, Y.))(\phi) = S(E_{\hat{\mu}}[\Phi])(\phi)$$

The injectivity of the S -transform yields

$$\pi(t, Y.) = E_{\hat{\mu}}[\Phi]. \quad (37)$$

Finally, by Itô's Lemma we get that $X_t = \Psi(Y.)(t)$. So choosing $\pi(t, \phi) = \Psi(\phi)(t)$ in (37) yields the result. ■

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